Eleven lines cannot form 33 Kobon triangles: a checked SAT proof over all pseudoline arrangements

Version 1, 2 November 2026

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Abstract

The Kobon triangle problem asks for the largest number $K(n)$ of bounded triangular faces in an arrangement of n lines in the plane. For $n = 11$ the best known arrangements have 32 triangles and the classical counting bound allows 33. In October 2026 the value $a(11)$ was removed from OEIS A006066, because the lemma that extends the counting bound to arrangements in which three or more lines meet at a point has a counterexample. We show that no arrangement of 11 pseudolines, simple or not, has 33 bounded triangles; hence $K(11) = 32$. A counting argument reduces every arrangement with at least $n(n-2)/3$ triangles to two cases: simple arrangements, and arrangements in which three concurrent lines carry three particular triangles. Each case is a SAT instance over rank-3 chirotopes. The simple case is refuted by a single LRAT proof checked with `lrat-check`. The other case is refuted by a cube-and-conquer tree with 67,614 leaves, each refuted by an LRAT proof that was checked as it was produced, and 1,094 splits whose covers were checked in the same way. The computation took about 2,540 CPU hours.

1 The question

For n lines in the real affine plane, let $K(n)$ be the largest number of bounded faces that are triangles. Honma and Kabanovitch found arrangements of 11 lines with 32 triangles [8], and Tamura's counting bound $\lfloor n(n-2)/3 \rfloor$, proved for simple arrangements, gives 33. For simple arrangements of 11 pseudolines meeting this bound, Savchuk [9] showed by SAT that 33 triangles are impossible. The reduction of the general case to simple arrangements rested on Lemmas 1 and 2 of Clément and Bader [4]. Parpalak and Utkin pointed out a counterexample to Step 3 of Lemma 1: in the arrangement formed by the sides and medians of a triangle, the centroid is a vertex of six pairs of triangles sharing a side [8]. On 9 October 2026 Savchuk removed $a(11)$ to $a(17)$ from OEIS A006066 and recorded this reason [8]. The draft of Clément and Bader also states that Tamura's bound "directly follows from the proof of Lemma 1" [4], so for arrangements with multiple points the bound itself was open.

Theorem 1. *No arrangement of* 11 *pseudolines in the affine plane has* 33 *or more bounded triangular faces. Hence* $K(11) = 32$.

Here an arrangement of pseudolines is the affine part of a projective arrangement of pseudolines, one of which is taken as the line at infinity (Section 2); every pseudoline meets the line at infinity once. Straight-line arrangements are of this kind, so the theorem covers them, simple or not. Families of curves that merely cross pairwise at most once are not covered. Section 2 reduces the theorem to two cases, Section 3 encodes each case as a SAT instance, and Section 4 reports the checked refutations. Section 6 lists what the proof rests on.

2 Segments, sectors and the case split

An affine arrangement of n pseudolines comes from a projective arrangement of $n+1$ pseudolines by declaring one of them the line at infinity; two lines are *parallel* when they meet on it. Parallelism is an equivalence relation, with classes of sizes q. A *vertex* is a point of the affine plane on at least two lines; its multiplicity m_v is the number of lines through it. A *segment* is a bounded piece of a line between two consecutive vertices on it. At a vertex v the m_v lines leave along $2m_v$ rays, which divide a small disc around v into $2m_v$ *sectors*; two sectors are *opposite* when they lie between the same two lines. A face with v on its boundary meets exactly one sector at v. A sector is *triangular* if the face it meets is a bounded triangle; then v is a corner of that triangle. Let T be the number of bounded triangular faces.

Lemma 2. *Each side of a bounded triangular face is a single segment, and each segment is a side of at most two such faces, one on each side of its line. If a segment PQ is a side of two of them, then $m_P \geq 3$ or $m_Q \geq 3$.*

Proof. A line through an interior point of a side crosses the side there and enters the open triangle, so no vertex lies inside a side. On each side of a line exactly one face is adjacent to a given segment. Let PQ lie on line L. A triangle on one side of L with side PQ is bounded by L, a line A through P and a line B through Q, with apex $A \cap B$ on that side; a triangle on the other side uses lines A' through P and B' through Q. If $A = A'$ and $B = B'$, then A and B meet on both sides of L, which two pseudolines cannot do. Hence $A \neq A'$ or $B \neq B'$, and P or Q lies on at least three lines. □

Lemma 3. *If $T > 0$, the number of segments is*

$$S \;=\; n(n-2) \;-\; \sum_{\text{parallel classes}} q(q-1) \;-\; \sum_{v} m_v(m_v-2).$$

Proof. If some line were parallel to all others, all lines would be pairwise parallel and $T = 0$. So every line L meets the $n-1-p_L$ lines not parallel to it, where p_L is the number of lines parallel to L. A vertex v on L accounts for $m_v - 1$ of them, so L carries $k_L = n-1-p_L-\sum_{v\in L}(m_v-2) \geq 1$ vertices and $k_L - 1$ segments. Summing over the lines, $\sum_L p_L = \sum q(q-1)$ and each vertex is counted on m_v lines. □

Lemma 4. *If a segment is a side of two triangular faces, then at each endpoint v of the segment, both sectors next to the ray of v along the segment are triangular.*

Proof. Each of the two triangles has v as a corner. Near v it fills the wedge between the ray along the segment and the ray of its other side through v, and since it is a face no further ray at v lies inside that wedge. So the wedge is a sector at v, and the two triangles fill the two sectors on either side of the ray. □

For a vertex v let d_v be the number of rays at v whose two neighbouring sectors are both triangular. Equivalently, d_v is the number of cyclically consecutive pairs of triangular sectors at v, and $d_v \leq 2m_v$.

Proposition 5. *Let an affine arrangement of $n \geq 3$ pseudolines have T bounded triangular faces with $3T \geq n(n-2)$. Then one of the following holds.*

(A) No two lines are parallel and no three lines pass through a common point.

(B) After relabelling, lines $1,2,3$ *pass through a common point; lines* $1,2,4$ *bound a triangular face and lines* $1,2,5$ *bound a triangular face; and lines* $2,3,z$ *bound a triangular face for some* $z \notin \{1,2,3\}$.

Proof. Since $3T \geq n(n-2) > 0$, Lemma 3 applies. By Lemma 2 every triangle has three sides, each a segment, and a segment is a side of at most two triangles, so $3T \leq S + D$, where D is the number of segments that are sides of two triangles. By Lemmas 2 and 4, each such segment has an endpoint v with $m_v \geq 3$, and its ray at v is one of the d_v rays counted at v; distinct segments give distinct rays. Hence $D \leq \sum_{m_v \geq 3} d_v$. With $3T \geq n(n-2)$ and Lemma 3,

$$\sum_{m_v \geq 3} g_v \;\geq \sum_{\text{parallel classes}} q(q-1) \;\geq\; 0, \qquad g_v = d_v - m_v(m_v-2),$$

because vertices with $m_v = 2$ contribute nothing to the sum in Lemma 3.

If no vertex has multiplicity at least 3, the left side is 0, so there are no parallel lines; three lines through a common point would be a vertex of multiplicity 3. This is case (A).

Otherwise the left side is a non-empty sum, so some vertex v with $m_v \geq 3$ has $g_v \geq 0$. Since $g_v \leq 2m_v - m_v(m_v-2) = m_v(4-m_v)$, either $m_v = 3$ and $d_v \geq 3$, or $m_v = 4$ and $d_v = 8$.

Let $m_v = 3$, with six sectors in cyclic order. If all six are triangular, any four consecutive ones are. Otherwise the triangular sectors form $k \geq 1$ maximal runs of lengths $r_1, \ldots, r_k$, separated by at least k non-triangular sectors, and $d_v = \sum_i (r_i - 1) \leq (6-k) - k$. From $d_v \geq 3$ we get $k = 1$ and $r_1 \geq 4$. So four consecutive sectors $s_1, \ldots, s_4$ are triangular. With three lines through v, s_1 and s_4 are opposite; call their two lines 1 and 2. The sector s_2 shares a ray with s_1; name the lines so that this ray lies on line 2. Then s_2 lies between line 2 and the third line through v, call it 3.

Let $m_v = 4$ and $d_v = 8$, so all eight sectors are triangular. Take any sector, between lines 1 and 2; its opposite sector is also triangular, and the sector next to it across the ray of line 2 lies between line 2 and a third line through v, call it 3.

In both cases lines $1,2,3$ pass through v. A triangular sector between lines 1 and 2 belongs to a triangle with sides on 1, 2 and a third line x, so lines $1,2,x$ bound a triangular face; the opposite sector gives such a line y. Three pairwise crossing pseudolines not through one point form an arrangement isomorphic to one of three lines, which has exactly one bounded face, lying in one of the four sectors formed at $1 \cap 2$ by lines 1 and 2. A triangular face with sides on $1,2,x$ is that face, so the two triangles in opposite sectors have $x \neq y$; relabel them 4 and 5. The triangular sector between lines 2 and 3 gives a triangle with sides on 2, 3 and some $z \notin \{1,2,3\}$. This is case (B). □

Remark 6. Proposition 5 does not prove Tamura's bound for arrangements with multiple points; it shows only that an arrangement at or above the bound is simple or contains the configuration of case (B). For $n = 11$, $3 \cdot 33 = 99 = n(n-2)$, so it applies.

3 The SAT model

Arrangements as oriented matroids. Label the line at infinity 0 and the lines $1, \ldots, n$. By the Folkman–Lawrence topological representation theorem [6, 2], projective arrangements of pseudolines correspond to simple rank-3 oriented matroids. We encode a chirotope χ on the triples of $\{0, \ldots, n\}$ with values in $\{+, -, 0\}$ and impose: χ is alternating; every pair of elements lies in a triple with $\chi \neq 0$; the zero triples form a simple rank-3 matroid (if *abc* and *abd* are dependent, so are *acd* and *bcd*); and the 3-term Grassmann–Plücker relations hold: for every a and $x_1, \ldots, x_4$ the products $\chi(a,x_1,x_2)\chi(a,x_3,x_4)$, $-\chi(a,x_1,x_3)\chi(a,x_2,x_4)$ and $\chi(a,x_1,x_4)\chi(a,x_2,x_3)$ are all zero or

include both signs. A support that is a matroid together with the 3-term relations characterises chirotopes [2, Thm. 3.6.2]. Lines a, b are parallel when $\chi(a, b, 0) = 0$, and lines a, b, c pass through a common point when $\chi(a, b, c) = 0$.

Bounded triangles. For non-parallel lines a, b, the vertex $a \cap b$ lies on side $s_{ab}(d) = \chi(a, b, d)\,\chi(a, b, 0)$ of a line d; the factor $\chi(a, b, 0)$ normalises the vertex by the line at infinity. If lines a, b, c bound a triangular face, they are pairwise non-parallel, $\chi(a, b, c) \neq 0$, and no other line d gives two of the three vertices opposite nonzero sides, since such a d would cross the open triangle.

Encoding. Variables encode χ, the vertex sides and an indicator T_{abc} for each triple of lines; T_{abc} implies the conditions above, and a sequential counter requires at least K indicators to be true. Only this direction is needed: an arrangement with K triangles, with T_{abc} set true exactly for its triangles, satisfies the formula. An unsatisfiable instance therefore excludes every arrangement of n pseudolines with at least K bounded triangles.

The two cases. Case A adds $\chi(a, b, c) \neq 0$ for all lines a, b, c. Case B adds $\chi(1, 2, 3) = 0$, T_{124}, T_{125} and the clause $\bigvee_{z \geq 4} T_{23z}$.

Symmetry breaking. Case A fixes T_{123}. Both cases fix the orientation by $\chi(0, 1, 2) = +$; $\chi(0, 1, e) \in \{+, 0\}$ for $e \geq 3$; $\chi(0, 2, e) = +$ when $\chi(0, 1, e) = 0$; and $\chi(1, 2, 3) = +$ in case A, $\chi(1, 2, 4) = +$ in case B. With the option `--lex`, the lines without a special role ($4, \ldots, n$ in case A, $6, \ldots, n$ in case B) are ordered: for every i in that range, $\chi \leq_{\text{lex}} \chi \circ (i\ i{+}1)$ over all triples in sorted order, with $- < 0 < +$.

Lemma 7. *Every arrangement in case A (resp. B) with at least $K \geq 1$ triangles has a labelling and orientation that satisfies the case A (resp. B) formula with all symmetry-breaking constraints.*

Proof. Label the lines as in Proposition 5 for case B, and so that $1, 2, 3$ bound a triangle for case A. Reorienting an element (reversing a pseudoline) changes the sign of χ on the triples containing it and does not change the arrangement, so it keeps triangles, parallelism and concurrency; so does multiplying χ by -1. Lines 1 and 2 are not parallel, so $\chi(0, 1, 2) \neq 0$; multiply by -1 if it is negative. For $e \geq 3$ with $\chi(0, 1, e) = -$, reverse e. If $\chi(0, 1, e) = 0$ then $\chi(0, 2, e) \neq 0$, since otherwise lines $1, 2, e$ would pass through one point at infinity and $1, 2$ would be parallel; reverse e if it is negative. Reversing all of $1, \ldots, n$ changes the sign of χ on triples of lines and keeps it on triples containing 0; do so if $\chi(1, 2, 3)$ (case A) or $\chi(1, 2, 4)$ (case B) is negative, both being nonzero because of T_{123} and T_{124}. All remaining constraints, including the counter and the case clauses, are invariant under permutations of $\{4, \ldots, n\}$ (case A) or $\{6, \ldots, n\}$ (case B), and the orientation constraints are stated alike for every $e \geq 3$. Among these relabellings the lexicographically smallest χ satisfies the `--lex` constraints. □

Redundant constraints in case A. In case A every segment is a side of at most one triangle (Lemma 2) and each line carries $n - 2$ segments. So each line a is a side of $t_a \leq n - 2$ triangles, and $\sum_a t_a = 3T \geq 3K$ gives $t_a \geq 3K - (n - 1)(n - 2)$. The option `--redundant` adds $t_a + p_a \leq n - 2$ (where p_a counts lines parallel to a, zero in case A) and the lower bound. For $n = 11$, $K = 33$ these force $t_a = 9$ for every line.

4 Computation

Case A. The instance `kobon_cnf.py 11 33 --simple --redundant --lex` has 33,953 variables and 227,427 clauses. CaDiCaL 3.0.1 [1] refuted it in 15,752 s on one core, writing an LRAT proof of 22.66 GB, which `lrat-check` from the DRAT-trim tools [10, 5] verified in 573 s.

Case B. The instance `kobon_cnf.py 11 33 --case-b --lex` has 31,212 variables and 196,620 clauses. We used cube and conquer [7]. The look-ahead solver `march_cu` split the instance at depth 8 into 238 cubes. Because `march_cu` drops cubes that it refutes itself, a split was accepted only after a *cover check*: the formula together with the negations of all its cubes was refuted by CaDiCaL with an LRAT proof verified by `lrat-check`. Each cube was then solved by CaDiCaL with the cube's literals as unit clauses, its LRAT proof streamed through a named pipe into `lrat-check`; a result was recorded as UNSAT only when the checker reported VERIFIED. No proof was stored. A cube not solved within a time limit was split again at depth 6 (at most 64 sub-cubes) into a child node whose cover was checked in the same way. The limit was 900 s in a first pass, then 1,800 s (for four hours on 28 October, 600 s), and none at the third level. Six cubes at the third level ran for hours; we split them again on idle cores and kept whichever refutation finished first. Five were closed by their direct run, after up to five hours, and one through two further levels of splitting.

The final tree has 1,094 checked splits, counting the top one, and 67,614 leaves, every one refuted by a verified LRAT proof; no cube is open and none is satisfiable. Leaves lie up to five levels below the top split. Solving the leaves took 1,997 CPU hours; runs stopped at the time limit before their cube was split took another 543. The work ran from 27 October 2026 on a ten-core laptop and from 1 November on AWS Graviton4 spot instances with 32 and then 192 cores; the cloud cost was about 22 US dollars.

5 Validation

The model reproduces known values over all arrangements. For $3 \le n \le 7$ it is satisfiable at $K = K(n)$ and unsatisfiable at $K(n)+1$, with $K(n) = 1, 2, 5, 7, 11$. For $n = 8$, $K = 16$ both cases are unsatisfiable with verified LRAT proofs, so $K(8) \le 15$ over all pseudoline arrangements, matching $K(8) = 15$. For $n = 9$, $K = 22$, simple arrangements are excluded by counting ($3 \cdot 22 > 63$), and case B is unsatisfiable with a verified LRAT proof of 3.84 GB, so $K(9) = 21$ over all pseudoline arrangements.

Independently of the model's face test, the script `validate_fixed.py` draws random straight-line arrangements with integer coefficients, including concurrent triples and parallel classes, counts their bounded triangles geometrically in exact rational arithmetic, fixes the model's chirotope to theirs, and checks that the formula without symmetry breaking is satisfiable at the true count and unsatisfiable one above it. In 60 arrangements of 4 to 8 lines, 50 non-simple and 39 with parallel lines, there was no mismatch.

6 Proof of Theorem 1 and what it rests on

Suppose an arrangement of 11 pseudolines has at least 33 bounded triangles. Since $3 \cdot 33 = 11 \cdot 9$, Proposition 5 puts it in case A or case B, and by Lemma 7 and the redundant constraints it then satisfies the corresponding formula with $K = 33$. Both formulas are unsatisfiable (Section 4). With the 32-triangle arrangements of Honma and Kabanovitch, $K(11) = 32$. □

The proof rests on the following, which a reader can check separately.

1. The hand arguments: Lemmas 2–4, Proposition 5 and Lemma 7. Proposition 5 replaces the step that failed in [4] and is the part we most want checked.

2. The model: the chirotope axioms, the face conditions and their encoding in `src/kobon_cnf.py`. That script regenerates both formulas byte for byte; their SHA-256 hashes are listed in the repository.

3. `lrat-check`, which verified every refutation. CaDiCaL and `march_cu` need not be trusted. `lrat-check` is not formally verified; the case-A proof is kept, so it and the covers can be re-checked with a verified checker such as `cake_lpr` [3].

4. The scripts that build the leaf and cover formulas and walk the tree (`run_cubes.py`, `check_cover.py`, `summary.py`). A cube counts as closed only if its latest record is a verified UNSAT, or if it was split with a verified cover and all its sub-cubes are closed.

The code, the cube files, all logs and the hashes are at `https://github.com/papexlabs/kobon-triangles`. The result records do not store a hash of the formula they refer to. The script `recheck.py` therefore rebuilds every node formula from the published top formula and the cubes on its path, re-solves chosen leaves with streamed LRAT checks and re-checks the covers on their paths. Leaf proofs were checked as they were produced and not stored, so re-checking all of case B means re-running the leaves, about 2,000 CPU hours; any subset can be re-run on its own. From the published files we re-solved 100 random leaves (with a recorded time of at most 300 s) and re-checked the 162 covers on their paths, including the top split. We also re-solved all 124 leaves of the subtree that was closed through extra splits, and re-checked the 6 covers on its path. All were verified.

Fingerprints. The SHA-256 hashes below fix the exact data this version refers to. All of it is in commit `300e0fe777d0a4291a4b02e990afe87f6f74e740` of the repository.

Case A formula	`f153fdb9b2c01d94fe8797b9dab9986296f3c9891e5f1607c5b2620f2f4d56d2`
Case B formula	`f6c143c271f7bce4ed9d65495e7e045c253b70401c50551b08e20a563a918d2b`
Cubes of the top split	`9b665d2d8813ef45d624b62bd14a1cd7adde290d1d9406037f840e8c145b34f9`
Cover check of the top split	`55895f383b9486f71b186f9fabf48f0257c169ebb789fb11b66dda836096758b`
Hashes of the case B tree's 3,298 files	`8209bd3441a4fe632bfffbe5a57662eff237ad4c8a6bea33c8dcb901182879e8`

The formulas are the output of `kobon_cnf.py` with the options `11 33 --simple --redundant --lex` and `11 33 --case-b --lex`. The other three files are `build/k11B_lex_d8.icnf`, `results/cubes_k11B_lex_d8.cover.out` and `results/cubes_k11B_lex_d8.sha256`.

How this work was done

The model, the code, the computation and this paper were prepared by Claude Opus 5.5, an AI model, working for Papex Labs; a person at Papex Labs reviews and signs each version. The code was written from the published statements of the sources, not from the tools of [9]. Solvers and tools: CaDiCaL 3.0.1, Kissat 4.0.4 (early runs), `march_cu`, `lrat-check`. Before release, a separate Claude Opus 5.5 session, given the code and notes but not our conclusions, re-derived the hand arguments, tested them on 615 random straight-line arrangements (many with concurrent triples, parallel lines or a triangle with its medians), checked the symmetry breaking on 533 arrangements brought into normal form, audited the whole tree and re-solved 24 leaves and 8 covers from scratch. It found no error. This does not replace review by a mathematician, and this version has not been peer reviewed.

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